

Generators and relations for the shallow mod 2 Hecke algebra in levels $\Gamma_0(3)$ and $\Gamma_0(5)$

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Abstract

Let $M(\text{odd}) \subset Z/2[[x]]$ be the space of odd mod 2 modular forms of level $\Gamma_0(3)$. It is known that the formal Hecke operators $T_p : Z/2[[x]] \rightarrow Z/2[[x]]$, p an odd prime other than 3, stabilize $M(\text{odd})$ and act locally nilpotently on it. So $M(\text{odd})$ is an $\mathcal{O} = Z/2[[t_5, t_7, t_{11}, t_{13}]]$ -module with t_p acting by T_p , $p \in \{5, 7, 11, 13\}$. We show:

- (1) Each $T_p : M(\text{odd}) \rightarrow M(\text{odd})$, $p \neq 3$, is multiplication by some u in the maximal ideal, m , of \mathcal{O} .
- (2) The kernel, I , of the action of \mathcal{O} on $M(\text{odd})$ is (A^2, AC, BC) where A, B, C have leading forms $t_5 + t_7 + t_{13}, t_7, t_{11}$.

We prove analogous results in level $\Gamma_0(5)$. Now \mathcal{O} is $Z/2[[t_3, t_7, t_{11}, t_{13}]]$, and the leading forms of A, B, C are $t_3 + t_7 + t_{11}, t_7, t_{13}$.

Let HE , “the shallow mod 2 Hecke algebra (of level $\Gamma_0(3)$ or $\Gamma_0(5)$)” be \mathcal{O}/I . (1) and (2) above show that HE is a 1 variable power series ring over the 1-dimensional local ring $Z/2[[A, B, C]]/(A^2, AC, BC)$. For another approach to all these results, based on deformation theory, see Deo and Medvedovsky [4].

1 Introduction

For p an odd prime, $T_p : Z/2[[x]] \rightarrow Z/2[[x]]$ is the formal Hecke operator $\sum c_n x^n \rightarrow \sum c_{pn} x^n + \sum c_n x^{pn}$; the T_p commute. We’ll be concerned with certain subspaces of $Z/2[[x]]$, coming from modular forms of level $\Gamma_0(N)$, and stabilized by the T_p , $p \nmid N$. On the spaces we’re looking at, each T_p acts locally nilpotently. Let S be a finite set of odd primes p not dividing N , and \mathcal{O} be a power series ring over $Z/2$ in variables t_p , $p \in S$. Then our subspace is an \mathcal{O} -module with t_p acting by T_p . We’ll show in some cases that each T_p acting on the subspace is multiplication by an element of \mathcal{O} (which lies in the maximal ideal since T_p acts locally nilpotently). And we’ll describe the kernel, I , of the action of \mathcal{O} on the subspace.

The motivating level 1 example appears in [3]. Let F in $Z/2[[x]]$ be $x + x^9 + x^{25} + \dots$, the exponents being the odd squares. The subspace V is spanned by the F^k , k odd (and positive). F is the mod 2 reduction of the weight 12 cusp form Δ , and a modular forms interpretation of V shows that the T_p stabilize it; with more work one may show that they act locally nilpotently. Take $S = \{3, 5\}$. Nicolas and Serre show:

- (1) Each $T_p : V \rightarrow V$ is multiplication by an element of $Z/2[[t_3, t_5]]$.
- (2) $\mathcal{O} = Z/2[[t_3, t_5]]$ acts faithfully on V .

Here is a level $\Gamma_0(3)$ example whose study was begun in [1]. Let $G = F(x^3)$, and $M(\text{odd})$ be spanned by the $F^i G^j$, where i, j are ≥ 0 and $i+j$ is odd. Here's a modular forms interpretation; $M(\text{odd})$ consists of all odd power series that are mod 2 reductions of elements of $Z[[x]]$ arising as expansions at infinity of holomorphic modular forms of level $\Gamma_0(3)$ (and any weight). (We write x in place of the customary q for the expansion variable throughout.) This interpretation shows that the T_p , $p \neq 3$, stabilize $M(\text{odd})$. Using the local nilpotence of the T_p acting on V , and on a certain subquotient W of $M(\text{odd})$ introduced in [1], we show that the T_p , $p \neq 3$, act locally nilpotently on $M(\text{odd})$.

If we take G to be $F(x^5)$ instead of $F(x^3)$ we get another subspace, which we'll also call $M(\text{odd})$; it has a similar interpretation with $\Gamma_0(3)$ replaced by $\Gamma_0(5)$. This $M(\text{odd})$ is stabilized by the T_p , $p \neq 5$, and we'll use results from [2] to show that they act locally nilpotently on it.

We take S to be $\{5, 7, 11, 13\}$ in the level 3 example and to be $\{3, 7, 11, 13\}$ in level 5. We will show that the T_p , $p \neq 3$, are multiplication by elements of \mathcal{O} in the first case, while the T_p , $p \neq 5$, are multiplication by elements of \mathcal{O} in the second. In each case we'll determine the kernel, I , of the action. It is an ideal (A^2, AC, BC) where the degree 1 parts of A , B and C are linearly independent in the 4-dimensional vector space m/m^2 . Apart from results from [1], [2], [3] there are 2 simple new ideas. One is making use of the fact that a certain \mathcal{O} -submodule of V imbeds in the subquotient, W , of $M(\text{odd})$. The other is showing that there are no non-zero \mathcal{O} -linear maps $W \rightarrow V$.

Shaunak Deo and Anna Medvedovsky, [4], have derived the same results simultaneously with us. They use techniques from deformation theory in place of our arguments, which are more related to ideas from [3]. Communications in both directions were helpful in understanding precisely what the kernel, I , of the action of \mathcal{O} on $M(\text{odd})$ should be, and in completing the proofs, both for us in our arguments and for them in theirs. It would be interesting to understand how the proofs are related.

2 $M(\text{odd})$ in level $\Gamma_0(3)$

Throughout this section $pr : Z/2[[x]] \rightarrow Z/2[[x]]$ is the map $\sum c_n x^n \rightarrow \sum_{(n,3)=1} c_n x^n$, G is $F(x^3)$ and $D = pr(F)$. (We'll use a related but different notation in the next section.)

Definition 2.1 $U_3 : Z/2[[x]] \rightarrow Z/2[[x]]$ is the map $\sum c_n x^n \rightarrow \sum c_{3n} x^n$. $M(\text{odd}) \subset Z/2[[x]]$ is spanned by the $F^i G^j$, $i, j \geq 0$, $i + j$ odd.

As was shown in [1], there is an interpretation of $M(\text{odd})$ in terms of modular forms of level $\Gamma_0(3)$ that shows that the T_p , $p \neq 3$, stabilize it. It's also stabilized by U_3 (by a similar argument) but we'll only need the obvious fact that U_3 maps the space spanned by the G^k , k odd, bijectively to V , and that this map commutes with T_p , $p \neq 3$. The following are proved in [1]:

Theorem 2.2

- (1) F has degree 4 over $Z/2(G)$ and $(F + G)^4 = FG$.
- (2) As $Z/2[G^2]$ -module, $M(\text{odd})$ has basis $\{G, F, F^2G, F^3\}$.
- (3) The trace map $Z/2(F, G) \rightarrow Z/2(G)$ takes G, F, F^2G, F^3 to $0, 0, 0, G$. So it gives a $Z/2[G^2]$ -linear map $Tr : M(\text{odd}) \rightarrow M(\text{odd})$. The kernel $N2$ and image $N1$ of Tr have $Z/2[G^2]$ -bases $\{G, F, F^2G\}$ and $\{G\}$.
- (4) The filtration $M(\text{odd}) \supset N2 \supset N1 \supset (0)$ of $M(\text{odd})$ is "Hecke-stable." I.e., the T_p , $p \neq 3$, stabilize $N2$ and $N1$.
- (5) The image, W , of $N2$ under pr has $Z/2[G^2]$ -basis $\{D, D^2G\}$. $N1$ is the kernel of $pr : N2 \rightarrow W$, and so the T_p , $p \neq 3$, stabilize W , and the isomorphism $N2/N1 \rightarrow W$ commutes with T_p , $p \neq 3$.

Remarks Besides the above bijection $N2/N1 \rightarrow W$ commuting with T_p , $p \neq 3$, we have the bijection $U_3 : N1 \rightarrow V$ commuting with these T_p . Finally there is a composite bijection $M(\text{odd})/N2 \xrightarrow{Tr} N1 \xrightarrow{U_3} V$. We'll show that this too commutes with T_p .

Lemma 2.3 $T_3 : V \rightarrow V$ is onto.

Proof By [3], V has a $Z/2$ -basis $\{m_{i,j}\}$ with $m_{0,0} = F$, "adapted to T_3 and T_5 ." But then T_3 takes $m_{i+1,j}$ to $m_{i,j}$. \square

Observation Here's an "injective modules are divisible" generalization of the above. Suppose M is a $k[[X, Y]]$ -module that admits a k -basis $m_{i,j}$ adapted to X and Y . Then if u in $k[[X, Y]]$ is non-zero, $uM = M$. (And in particular, the action is faithful.) To see this, totally order $N \times N$, taking $(0, 0) < (1, 0) < (0, 1) < (2, 0) < (1, 1) < (0, 2) < (3, 0) < \dots$. Let $cX^a Y^b$ be the monomial appearing in the leading form of u with largest a . Then $u(m_{a+i, b+j}) = cm_{i,j} +$ a k -linear combination of $m_{r,s}$ with $(r, s) < (i, j)$, and an inductive argument using the total ordering gives the result.

Lemma 2.4 *The composite map $V \xrightarrow{T_r} N1 \xrightarrow{U_3} V$ is T_3 , and so, by Lemma 2.3, is onto.*

Proof Let $U(X, Y)$ be $(X + Y)^4 + XY$. Then $U(F(x), F(x^3)) = 0$. Replacing x by x^3 we find that $U(G, F(x^9)) = 0$. So $U(F(x^9), G) = 0$, and $F(x^9)$ is a conjugate of F over $Z/2(G)$. Similarly, replacing x by rx where r is in the field of 4 elements, $r^3 = 1$, we find that the other 3 conjugates of F are the $F(rx)$. So the conjugates of F^k are $F^k(x^9)$ and $F^k(rx)$. Writing F^k as $\sum c_n x^n$, adding together the conjugates, and applying U_3 we get $T_3(F^k)$. \square

Theorem 2.5 *The composite bijection $M(\text{odd})/N2 \xrightarrow{T_r} N1 \xrightarrow{U_3} V$ commutes with T_p , $p \neq 3$. We conclude that $M(\text{odd})/N2$, $N2/N1$ and $N1$ identify with V , W and V as Hecke-modules.*

Proof By Lemma 2.4 the restriction of our map to V is onto. So the elements of V span $M(\text{odd})/N2$. And on V our map is $T_3 : V \rightarrow V$ which commutes with the $T_p : V \rightarrow V$. \square

Theorem 2.6 *The T_p , $p \neq 3$, act locally nilpotently on $M(\text{odd})$. In other words, if $f \in M(\text{odd})$ and $p \neq 3$, some power of T_p annihilates f .*

Proof [3] and [1] show that T_p acts locally nilpotently on V and W . And the quotients in the filtration of $M(\text{odd})$ are Hecke-isomorphic to V , W and V . \square

For the rest of this section, unless otherwise noted, $S = \{5, 7, 11, 13\}$ and \mathcal{O} is the 4-variable power series ring over $Z/2$ in the t_p , $p \in S$. Then V , W and $M(\text{odd})$ are all \mathcal{O} -modules with t_p acting by T_p , $p \in S$. Let $I(V)$, $I(W)$ and I be the kernels of the respective actions.

$I(V)$ is easily described. As we noted in section 1, when V is viewed as a $Z/2[[t_3, t_5]]$ -module, the action is faithful, and each $T_p : V \rightarrow V$ is multiplication by some u in $Z/2[[t_3, t_5]]$. In [3] it's shown that:

$$\begin{aligned} \text{when } p = 11 \quad u &= \text{unit}(t_3) \\ \text{when } p = 13 \quad u &= \text{unit}(t_5) \\ \text{when } p = 7 \quad u &= \text{unit}(t_3)(t_5) \end{aligned}$$

It follows from the above that when V is viewed as a $Z/2[[t_{11}, t_{13}]]$ -module the action is faithful, and each $T_p : V \rightarrow V$ is multiplication by some u in $Z/2[[t_{11}, t_{13}]]$. Furthermore when $p = 5$, $u = \text{unit}(t_{13})$, while when $p = 7$, $u = \text{unit}(t_{11})(t_{13})$. So $I(V)$ is generated by 2 elements, $t_5 + \text{unit}(t_{13})$ and $t_7 + \text{unit}(t_{11})(t_{13})$. This gives:

Theorem 2.7 *$I(V)$ is generated by 2 elements A and B whose leading forms can be taken to be $t_5 + t_7 + t_{13}$ and t_7 .*

To describe $I(W)$ we use the following results from [1]:

Theorem 2.8 *Let $W1$ and $W5$ be the $Z/2[G^2]$ -submodules of W generated by D and D^2G respectively. (In fact $G = D^3$, so that a $Z/2$ -basis of $W1$ consists of the D^k , $k \equiv 1 \pmod{6}$, while a $Z/2$ -basis of $W5$ consists of the D^k , $k \equiv 5 \pmod{6}$.)*

- (1) *The T_p , $p \equiv 1 \pmod{6}$, stabilize $W1$ and $W5$. The T_p , $p \equiv 5 \pmod{6}$, map $W1$ to $W5$, and $W5$ to $W1$.*
- (2) *$W1$ has a basis $\{m_{i,j}\}$ with $m_{0,0} = D$, adapted to T_7 and T_{13} . The same holds for $W5$ with $m_{0,0} = D^5 = D^2G$. It follows that T_7 and T_{13} act locally nilpotently on $W1$, $W5$ and W .*
- (3) *Taking $S = \{7, 13\}$ and making W into a $Z/2[[t_7, t_{13}]]$ -module, we find that each $T_p : W \rightarrow W$, $p \equiv 1 \pmod{6}$, is multiplication by some u in the maximal ideal of $Z/2[[t_7, t_{13}]]$.*
- (4) *And each $T_p : W \rightarrow W$, $p \equiv 5 \pmod{6}$, is the composition of T_5 with multiplication by some u in $Z/2[[t_7, t_{13}]]$.*
- (5) *$T_5^2 : W \rightarrow W$ is multiplication by λ^2 for some λ in $Z/2[[t_7, t_{13}]]$ with leading form $t_7 + t_{13}$.*

Remarks (3), (4) and (5) show that each T_p , $p \neq 3$, acts locally nilpotently on W . And if we take $S = \{5, 7, 13\}$, each $T_p : W \rightarrow W$, $p \neq 3$, is multiplication by an element of $Z/2[[t_5, t_7, t_{13}]]$. Furthermore if we set $\varepsilon = t_5 + \lambda$, then ε^2 kills W . Note that the leading form of ε is $t_5 + t_7 + t_{13}$.

Theorem 2.9 *$I(W)$ is generated by 2 elements ε^2 and C where ε is congruent to the A of Theorem 2.7 mod m^2 , and the leading form of C is t_{11} .*

Proof First we determine the kernel of the action of $Z/2[[t_5, t_7, t_{13}]]$ on W . The remark above shows that the kernel contains $(\varepsilon^2) = (t_5^2 + \lambda^2)$. If R is in the kernel, we may modify R by an element of this ideal, and assume that $R = u + u't_5$ with u and u' in $Z/2[[t_7, t_{13}]]$. Since u stabilizes $W1$ and $W5$ while $u't_5$ takes $W1$ to $W5$ and $W5$ to $W1$, u and $\lambda^2 u'$ are elements of $Z/2[[t_7, t_{13}]]$ annihilating W . Since $Z/2[[t_7, t_{13}]]$ acts faithfully on W (see for example the observation following Lemma 2.3), $u = u' = 0$, and the kernel of the action is (ε^2) ; note that ε has the same leading form as A . Finally by (4) of Theorem 2.8, $I(W)$ contains an element of the form $t_{11} + vt_5$ with v in \mathcal{O} . It remains to show that v is not a unit. But an easy calculation shows that $T_{11}(D^5) = 0$ while $T_5(D^5) = D$. \square

Definition 2.10 $V(0, \star)$ is the kernel of $T_3 : V \rightarrow V$.

If $\{m_{i,j}\}$ is a basis of V adapted to T_3 and T_5 , the $m_{0,j}$ form a $Z/2$ -basis of $V(0, \star)$. $Z/2[[t_5]]$ acts faithfully and locally nilpotently on $V(0, \star)$. $V(0, \star)$ is an \mathcal{O} -submodule of V , and the kernel of the action is a height 3 prime ideal P generated by t_7 , t_{11} and an element with leading form $t_{13} + t_5$.

Lemma 2.11 *W , as well as V , contains a “Hecke-submodule” isomorphic to $V(0, \star)$.*

Proof If $f \in V(0, \star)$, $T_3(f) = U_3(\text{Tr}(f)) = 0$. Since $U_3 : N1 \rightarrow V$ is bijective, $\text{Tr}(f) = 0$ and f is in $N2$. So we have a composite map $V(0, \star) \subset N2 \xrightarrow{pr} W$ which commutes with T_p , $p \neq 3$, and takes $m_{0,0} = F$ to D . Since $m_{0,0}$ is not in the kernel of this map, the kernel is (0) , giving the result. \square

Theorem 2.12 *Let A, B, C be as in Theorems 2.7 and 2.9. Then $(A, B, C) = I(V) + I(W) = P$, the kernel of the action of \mathcal{O} on $V(0, \star)$.*

Proof Evidently $(A, B, C) \subset I(V) + I(W)$. Since V and W each have an \mathcal{O} -submodule isomorphic to $V(0, \star)$, $I(V) + I(W) \subset P$. Finally (A, B, C) and P are height 3 primes of \mathcal{O} . \square

Theorem 2.13 *The only \mathcal{O} -linear map $\alpha : W \rightarrow V$ is the zero-map.*

Proof $\alpha(W)$ is annihilated both by $I(W)$ and $I(V)$. So by Theorem 2.12 it is annihilated by P , and thus by t_7 . Then $\alpha(t_7 W) = t_7 \alpha(W) = (0)$. But since $W1$ and $W5$ have bases adapted to T_7 and T_{13} , $t_7(W) = W$, and $\alpha(W) = (0)$. \square

Theorem 2.14 *If $p \neq 3$, $T_p : M(\text{odd}) \rightarrow M(\text{odd})$ is multiplication by some u in \mathcal{O} .*

Proof We know that $T_p : V \rightarrow V$ and $W \rightarrow W$ are multiplication by some u and u' in \mathcal{O} ; for W see the remarks following Theorem 2.8. Then $f \rightarrow T_p(f) - uf$ and $f \rightarrow T_p(f) - u'f$ both annihilate $V(0, \star)$ by Lemma 2.11. So $u - u'$ is in P , and by Theorem 2.12 it is in (A, B, C) . Modifying u by an element of (A, B) , and u' by an element of (C) we may assume $u - u' = 0$. Let $\alpha : M(\text{odd}) \rightarrow M(\text{odd})$ be the map $f \rightarrow T_p(f) - uf$. We'll show that α is the zero-map.

α annihilates V , and since $u = u'$, it annihilates W . Since α commutes with U_3 and pr it annihilates $N1$ and $N2/N1$. So $\alpha(N2) \subset N1$, and α induces an \mathcal{O} -linear map $N2/N1 \rightarrow N1$. By Theorem 2.13 this is the zero-map, and $\alpha(N2) = (0)$. But the proof of Theorem 2.5 shows that the elements of V span $M(\text{odd})/N2$. Since $\alpha(V) = 0$, $\alpha = 0$. \square

Theorem 2.15 *There are A, B, C in \mathcal{O} with leading forms $t_5 + t_7 + t_{13}$, t_7 and t_{11} such that $I(V) = (A, B)$ and $I(W) = (A^2, C)$. The kernel, I , of the action of \mathcal{O} on $M(\text{odd})$ is $I(V) \cap I(W) = (A^2, AC, BC)$.*

Proof Let A, B, C, ε be as in Theorems 2.7 and 2.9; $I(V) = (A, B)$, $I(W) = (\varepsilon^2, C)$ and $A - \varepsilon$ is in m^2 . Then $A^2 - \varepsilon^2$ is in $I(V) + I(W)$ which is (A, B, C) by Theorem 2.12. Since (A, B, C) is prime, $A - \varepsilon$ is in $(A, B, C) \cap m^2 = mA + mB + mC$. Modifying A by an element of $mA + mB$, and ε by an element of mC we may assume $A - \varepsilon = 0$. Then $I(V) = (A, B)$, $I(W) = (A^2, C)$ and it follows that $I(V) \cap I(W) = (A^2, AC, BC)$. Suppose u is in $I(V) \cap I(W)$. Let

$\alpha : M(\text{odd}) \rightarrow M(\text{odd})$ be multiplication by u . Then $\alpha(N1) = (0)$, $\alpha(N2) \subset N1$, and arguing as in the final paragraph of the proof of Theorem 2.14 we get the result; u is in I . \square

3 $M(\text{odd})$ in level $\Gamma_0(5)$

Throughout this section $pr : Z/2[[x]] \rightarrow Z/2[[x]]$ is the map $\sum c_n x^n \rightarrow \sum_{(n,5)=1} c_n x^n$, G is $F(x^5)$ and $D = pr(F)$.

Definition 3.1 $U_5 : Z/2[[x]] \rightarrow Z/2[[x]]$ is the map $\sum c_n x^n \rightarrow \sum c_{5n} x^n$. $M(\text{odd}) \subset Z/2[[x]]$ is spanned by the $F^i G^j$, $i, j \geq 0$, $i + j$ odd.

As was shown in [2], there is an interpretation of $M(\text{odd})$ in terms of modular forms of level $\Gamma_0(5)$ that shows that the T_p , $p \neq 5$, stabilize it. It is also stabilized by U_5 , but we'll only need the obvious fact that U_5 maps the space spanned by the G^k , k odd, bijectively to V , and that this map commutes with T_p , $p \neq 5$. The following are proved in [2]:

Theorem 3.2

- (1) F has degree 6 over $Z/2(G)$ and $(F + G)^6 = FG$.
- (2) As $Z/2[G^2]$ -module, $M(\text{odd})$ has basis $\{G, F, F^2G, F^3, F^4G, F^5\}$.
- (3) The trace map $Z/2(F, G) \rightarrow Z/2(G)$ takes $G, F, F^2G, F^3, F^4G, F^5$ to $0, 0, 0, 0, 0, G$. So it gives a $Z/2[G^2]$ -linear map $Tr : M(\text{odd}) \rightarrow M(\text{odd})$. The kernel $N2$ and image $N1$ of Tr have $Z/2[G^2]$ -bases $\{G, F, F^2G, F^3, F^4G\}$ and $\{G\}$.
- (4) The filtration $M(\text{odd}) \supset N2 \supset N1 \supset (0)$ of $M(\text{odd})$ is "Hecke-stable." I.e., the T_p , $p \neq 5$, stabilize $N2$ and $N1$.
- (5) The image, W , of $N2$ under pr has $Z/2[G^2]$ -basis $\{D, D^8/G, D^2G, D^4G\}$. $N1$ is the kernel of $pr : N2 \rightarrow W$, and so the T_p , $p \neq 5$, stabilize W , and the isomorphism $N2/N1 \rightarrow W$ commutes with T_p , $p \neq 5$.

Remarks Note that pr takes the elements F , F^2G and F^4G of $N2$ to D , D^2G and D^4G . Also $F(F + G)^2 = F(F + G)^8/FG = (F^8/G) + G^7$; pr takes this to D^8/G . Besides the bijection $N2/N1 \rightarrow W$ commuting with T_p , $p \neq 5$, we have the bijection $U_5 : N1 \rightarrow V$ commuting with these T_p . Finally there is a composite bijection $M(\text{odd})/N2 \xrightarrow{Tr} N1 \xrightarrow{U_5} V$; we'll show that this too commutes with T_p .

Lemma 3.3 $T_5 : V \rightarrow V$ is onto.

Proof If $m_{i,j}$ are as in Lemma 2.3, T_5 takes $m_{i,j+1}$ to $m_{i,j}$. \square

Lemma 3.4 *The composite map $V \xrightarrow{Tr} N1 \xrightarrow{U_5} V$ is T_5 , and so by Lemma 3.3, is onto.*

Proof We argue as in Lemma 2.4. Now, however, U is $(X + Y)^6 + XY$, and the conjugates of F over $Z/2(G)$ are $F(x^{25})$ and the $F(rx)$ where $r^5 = 1$. The argument is otherwise unchanged. \square

Theorem 3.5 *The composite bijection $M(\text{odd})/N2 \xrightarrow{Tr} N1 \xrightarrow{U_5} V$ commutes with T_p , $p \neq 5$. Together with the remarks following Theorem 3.2, this shows that $M(\text{odd})/N2$, $N2/N1$ and $N1$ identify with V , W and V as Hecke-modules.*

Proof See the proof of Theorem 2.5. \square

Theorem 3.6 *The T_p , $p \neq 5$, act locally nilpotently on $M(\text{odd})$. In other words, if $f \in M(\text{odd})$ and $p \neq 5$, some power of T_p annihilates f .*

Proof [3] and [2] show that T_p acts locally nilpotently on V and W . And the quotients in the filtration of $M(\text{odd})$ are Hecke-isomorphic to V , W and V . \square

For the rest of this section, unless otherwise noted, $S = \{3, 7, 11, 13\}$ and \mathcal{O} is the 4-variable power series ring over $Z/2$ in the t_p , $p \in S$. Then V , W and $M(\text{odd})$ are all \mathcal{O} -modules with t_p acting by T_p , $p \in S$. Let $I(V)$, $I(W)$ and I be the kernels of the respective actions.

$I(V)$ is easily described. As in the paragraph before Theorem 2.7 we view V as a $Z/2[[t_{11}, t_{13}]]$ -module. When $p = 3$, $T_p : V \rightarrow V$ is multiplication by $\text{unit}(t_{11})$, while when $p = 7$, T_p is multiplication by $\text{unit}(t_{11})(t_{13})$. So $I(V)$ is generated by two elements $t_3 + \text{unit}(t_{11})$ and $t_7 + \text{unit}(t_{11})(t_{13})$, and:

Theorem 3.7 *$I(V)$ is generated by 2 elements A and B whose leading forms can be taken to be $t_3 + t_7 + t_{11}$ and t_7 .*

To describe $I(W)$ we use the following results from [2]:

Theorem 3.8 *Let D_1, D_3, D_7, D_9 be $D, D^8/G, D^2G$ and D^4G . Define D_k for k positive and prime to 10 by $D_{k+10} = G^2 D_k$. Let W_a be spanned by D_k , $k \equiv 1, 3, 7, 9 \pmod{20}$, and W_b be spanned by D_k , $k \equiv 11, 13, 17, 19 \pmod{20}$. Then $W = W_a \oplus W_b$ and:*

- (1) *The T_p , $p \equiv 1, 3, 7, 9 \pmod{20}$, stabilize W_a and W_b . The T_p , $p \equiv 11, 13, 17, 19 \pmod{20}$, map W_a to W_b and W_b to W_a .*
- (2) *W_a has a basis $\{m_{i,j}\}$ with $m_{0,0} = D$ adapted to T_3 and T_7 . The same holds for W_b with $m_{0,0} = D_{11}$. It follows that T_3 and T_7 act locally nilpotently on W_a , W_b and W .*
- (3) *Taking $S = \{3, 7\}$ and making W into a $Z/2[[t_3, t_7]]$ -module, we find that each $T_p : W \rightarrow W$, $p \equiv 1, 3, 7, 9 \pmod{20}$, is multiplication by some u in the*

maximal ideal of $Z/2[[t_3, t_7]]$.

- (4) And each $T_p : W \rightarrow W$, $p \equiv 11, 13, 17, 19 \pmod{20}$ is the composition of T_{11} with multiplication by some u in $Z/2[[t_3, t_7]]$.
- (5) $T_{11}^2 : W \rightarrow W$ is multiplication by λ^2 for some λ in $Z/2[[t_3, t_7]]$ with leading form $t_3 + t_7$.

Remarks Now each $T_p : W \rightarrow W$, $p \neq 5$, is locally nilpotent on W . And if $S = \{3, 7, 11\}$, each T_p is multiplication by an element of $Z/2[[t_3, t_7, t_{11}]]$. And if we set $\varepsilon = t_{11} + \lambda$, then ε^2 kills W . Note that the leading form of ε is $t_3 + t_7 + t_{11}$.

Theorem 3.9 $I(W)$ is generated by 2 elements ε^2 and C , where ε is congruent mod m^2 to the A of Theorem 3.7 and the leading form of C is t_{13} .

Proof The proof is essentially the same as that of Theorem 2.9. At the very end we use (4) of Theorem 3.8 to see that $I(W)$ contains an element of the form $t_{13} + vt_{11}$ with v in \mathcal{O} . But an easy calculation shows that $T_{13}(D_{11}) = T_{13}(DG^2) = 0$ while $T_{11}(D_{11}) = T_{11}(x^{11} + \dots) = x + \dots \neq 0$. So v is not a unit, and our element has leading form t_{13} . \square

Definition 3.10 $V(\star, 0)$ is the kernel of $T_5 : V \rightarrow V$.

If $\{m_{i,j}\}$ is a basis of V adapted to T_3 and T_5 , the $m_{i,0}$ form a $Z/2$ -basis of $V(\star, 0)$. $Z/2[[t_3]]$ acts faithfully and locally nilpotently on $V(\star, 0)$. $V(\star, 0)$ is an \mathcal{O} -submodule of V , and the kernel of the action is a height 3 prime ideal, P , generated by t_7 , t_{13} and an element with leading form $t_{11} + t_3$.

Lemma 3.11 W , as well as V , contains a ‘‘Hecke-submodule’’ isomorphic to $V(\star, 0)$.

Proof If $f \in V(\star, 0)$, $T_5(f) = U_5(\text{Tr}(f)) = 0$. As in the proof of Lemma 2.11 we conclude that f is in N_2 and we get a composite map $V(\star, 0) \subset N_2 \xrightarrow{pr} W$ taking $m_{0,0}$ to D , which is the desired imbedding. \square

Theorem 3.12 Let A, B, C be as in Theorems 3.7 and 3.9. Then $(A, B, C) = I(V) + I(W) = P$, the kernel of the action of \mathcal{O} on $V(\star, 0)$.

The proof mimics that of Theorem 2.12.

Theorem 3.13 The only \mathcal{O} -linear map $\alpha : W \rightarrow V$ is the zero-map.

The proof is like that of Theorem 2.13, but this time we use the fact that W_a and W_b have bases adapted to T_3 and T_7 to see that $T_7(W) = W$.

Theorem 3.14 If $p \neq 5$, $T_p : M(\text{odd}) \rightarrow M(\text{odd})$ is multiplication by some u in \mathcal{O} .

We argue as in the proof of Theorem 2.14, using $V(\star, 0)$ in place of $V(0, \star)$.

Theorem 3.15 *There are A, B, C in \mathcal{O} with leading forms $t_3 + t_7 + t_{11}$, t_7 and t_{13} such that $I(V) = (A, B)$ and $I(W) = (A^2, C)$. The kernel, I , of the action of \mathcal{O} on $M(\text{odd})$ is $I(V) \cap I(W) = (A^2, AC, BC)$.*

The proof mimics that of Theorem 2.15.

References

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